

PALINDROMIC WIDTH OF FREE NILPOTENT GROUPS

VALERIY. G. BARDAKOV AND KRISHNENDU GONGOPADHYAY

ABSTRACT. In this paper we consider the palindromic width in free nilpotent groups. In particular, we prove that the palindromic width of finitely generated free nilpotent group is finite. For low rank and low step nilpotents, we provide precise estimates of the palindromic width.

1. INTRODUCTION

Let G be a group with a specified set of generators A . A reduced word in the alphabets $A^{\pm 1}$ is a *palindrome* if it reads the same forwards and backwards. The palindromic length $l_P(g)$ of an element g in G is the minimum number k such that g can be expressed as a product of k palindromes. The *palindromic width* of G is defined to be $pw(G) = \sup_{g \in G} l_P(g)$. In analogy with commutator width in groups (for example see [2, 3, 4, 5]), it is a problem of potential interest to study palindromic width in groups. Palindromes of free groups have already proved useful in studying various aspects of combinatorial group theory, for example see [8, 9, 13]. In [6], it was proved that the palindromic width of non-abelian free group is infinite. This result was generalized in [7] where the authors proved that almost all free products have infinite palindromic width; the only exception is given by the free product of two cyclic groups of order two, when the palindromic width is two. Piggot [15] studied the relationship between primitive words and palindromes in free groups of rank two. It follows from [6, 15] that up to conjugacy, a primitive word can always be written as either a palindrome or a product of two palindromes and that certain pairs of palindromes will generate the group. Recently Gilman-Keen [10, 11] have used tools from hyperbolic geometry to reprove this result and further have obtained discreteness criteria for two generator subgroups in $PSL(2, \mathbb{C})$ using the geometry of palindromes. The work of Gilman-Keen indicates deep connection between palindromic width in groups and geometry.

Let $N_{n,r}$ be the free nilpotent group of n generators and step r . In this paper we consider the palindromic width in free nilpotent groups. We prove that the palindromic width of finitely generated free nilpotent group is finite. In fact, we prove that the palindromic width of an arbitrary rank n free nilpotent group is bounded by $3n$. We further improve this bound for 2-step free nilpotent groups and the group $N_{2,3}$. For the groups, $N_{n,1}$ and $N_{2,2}$ we get the exact value of the palindromic width. Our main theorem is the following.

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Theorem 1.1. *Let $N_{n,r}$ be the free nilpotent group of n generators and step r . The following holds.*

- (1) *The palindromic width $pw(N_{n,1})$ of a free abelian group of n generators is equal to n .*
- (2) *For any positive integers $n \geq 2$ and $r \geq 1$, $n \leq pw(N_{n,r}) \leq 3n$.*
- (3) *For any $n \geq 2$, $pw(N_{n,2}) \leq 3(n-1)$. Further, $pw(N_{2,2}) = 3$ and $4 \leq pw(N_{3,2}) \leq 6$.*
- (4) *$3 \leq pw(N_{2,3}) \leq 6$.*

We prove the theorem in section 3. In section 2, after recalling some basic notions and related basic results, we prove Lemma 2.3 which is a key ingredient in the proof of Theorem 1.1. Another key ingredient in proving the bound for $N_{3,2}$ is the calculation of $pw(\overline{N}_{3,2})$, where $\overline{N}_{3,2}$ is the quotient group $\overline{N}_{3,2} = N_{3,2}/\langle x_1^2, x_2^2, x_3^2 \rangle$. In section 3.4 we prove the following theorem.

Theorem 3.1. *The palindromic width of the group $\overline{N}_{3,2}$ is 4.*

Motivated by the analysis of $pw(\overline{N}_{3,2})$, we pose the following problem that is natural to ask.

Problem 1.1. *Let $\overline{N}_{n,r} = N_{n,r}/\langle x_1^2, x_2^2, \dots, x_n^2 \rangle$. What can we say about $pw(\overline{N}_{n,r})$?*

It would be interesting to obtain the precise values of the palindromic width of $N_{n,r}$ for $n \geq 3$, $r \geq 2$.

- Problem 1.2.** (1) *For $n \geq 3$, $r \geq 2$, find $pw(N_{n,r})$.*
 (2) *Construct an algorithm that determine $l_{\mathcal{P}}(g)$ for arbitrary $g \in N_{n,r}$.*

The above problem can be asked for any other groups as well. In general, the palindromic width of an arbitrary group depends on the generating set of the group. However, the advantage of working with free nilpotent groups is that, in this case we have a free set of generators and hence, the palindromic width is independent of the choice of the generators.

Problem 1.3. *Is it true that for a finitely generated group $G = \langle A \rangle$ the palindromic width $pw(G)$ is finite if and only if its commutator width $cw(G)$ is finite?*

2. BACKGROUND AND PRELIMINARY RESULTS

2.1. Background.

2.1.1. Width in groups. Let G be a group and $A \subseteq G$ a subset that generates G . For each $g \in G$ define the *length* $l_A(g)$ of g with respect to A to be the minimal k such that g is a product of k elements of $A^{\pm 1}$. The supremum of the values $l_A(g)$, $g \in G$, is called the *width* of G with respect to A and is denoted by $wid(G, A)$. In particular, $wid(G, A)$ is either a natural number or ∞ . If $wid(G, A)$ is a natural number, then every element of G is a product of at most $wid(G, A)$ elements of A .

For g, h in G , the *commutator* of g and h is defined as $[g, h] = g^{-1}h^{-1}gh$. If \mathcal{C} is the set of commutators in some group G then the commutator subgroup G' is generated by \mathcal{C} . The length $l_{\mathcal{C}}(g)$ of an element $g \in G'$ is called the *commutator length* and is denoted by $cl(g)$. The width $wid(G', \mathcal{C})$ is called the *commutator width* of G and is denoted by $cw(G)$. It is well known [1] that the commutator width of a free non-abelian

group is infinite, but the commutator width of a finitely generated nilpotent group is finite (see [3, 4]). An algorithm of the computation of the commutator length in free non-abelian groups can be found in [5].

Let A be a set of generators of a group G . A reduced word w in the alphabets $A^{\pm 1}$ is called a *palindrome* if w reads the same left-to-right and right-to-left. An element g of G is called a *palindrome* if g can be represented by some word w that is a palindrome in the alphabets $A^{\pm 1}$. We denote the set of all palindromes in G by $\mathcal{P} = \mathcal{P}(G)$. Evidently, the set \mathcal{P} generates G . Then any element $g \in G$ is a product of palindromes

$$g = p_1 p_2 \dots p_k.$$

The minimal k with this property is called the *palindromic length* of g and is denoted by $l_{\mathcal{P}}(g)$. The *palindromic width* is given by

$$pw(G) = wid(G, \mathcal{P}) = \sup_{g \in G} l_{\mathcal{P}}(g).$$

2.1.2. Free Nilpotent Groups. Let $N_{n,r}$ be a free r -step nilpotent group of rank n with the generators x_1, \dots, x_n . For example, when $r = 1$, $N_{n,1}$ is simply the free abelian group generated by x_1, \dots, x_n , so every element of $N_{n,1}$ can be described uniquely as

$$g = x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

for some integers $\alpha_1, \dots, \alpha_n$. For $r = 2$, every element $g \in N_{n,2}$ has the form

$$(2.1) \quad g = \prod_{i=1}^n x_i^{\alpha_i} \cdot \prod_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}}$$

for some integers α_i and β_{ij} , where $[x_i, x_j] = x_i^{-1} x_j^{-1} x_i x_j$ are basis commutators (see [17, Chapter 5]). In the general case the following theorem hold (see [12, p. 175, Theorem 11.2.4], [17]).

Theorem 2.1. *If $N_{n,r}$ is the free r -step nilpotent group with free generators x_1, \dots, x_n and if in a sequence of basic commutators c_1, \dots, c_t are those of weights $1, 2, \dots, r$, then an arbitrary element $f \in N_{n,r}$ has a unique representation,*

$$f = c_1^{e_1} c_2^{e_2} \dots c_t^{e_t}.$$

For a free nilpotent group $N_{n,r}$, let $N'_{n,r}$ be its commutator subgroup. We note the following lemma that will be used later. This lemma was proved in the paper of Allambergenov and Roman'kov [4] also the prove of it can be find in [2]

Lemma 2.1. [2] *Any element g in the commutator subgroup $N'_{n,r}$ can be represented in the form*

$$g = [u_1, x_1] [u_2, x_2] \dots [u_n, x_n], \quad u_i \in N_{n,r}.$$

In fact, Allambergenov and Roman'kov [4] proved the following.

- (i) Any element of the commutator subgroup $N'_{n,2}$ is a product of no more than $[n/2]$ commutators, where $[a]$ is the integer part of a ;
- (ii) Any element of the commutator subgroup $N'_{n,r}$ in all other cases ($r \geq 3$, $n \geq 4$ or $r > 3$, $n = 2$) is a product of no more than n commutators.

Finishing the remaining case of $n = 2$, $r = 3$, Rhemtulla and Akhavan-Malayeri showed in [2] that any element of the commutator subgroup $N'_{2,3}$ is a product of two commutators.

2.2. Preliminary Results. Let $G = \langle A \rangle$ be a group and $\mathcal{P} = \mathcal{P}(A)$ be the set of palindromes in G . Evidently, that any palindrome $p \in \mathcal{P}$ can be presented in the form

$$p = ua^\alpha \bar{u}, \text{ for some } a \in A, \alpha \in \mathbb{Z},$$

where

$$u = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k}, \quad a_i \in A, \alpha_i \in \mathbb{Z}$$

is a word and

$$\bar{u} = a_k^{\alpha_k} a_{k-1}^{\alpha_{k-1}} \dots a_1^{\alpha_1}$$

is its reverse word. Clearly, $\bar{u}^{-1} = \overline{u^{-1}}$.

We note the following basic lemmas which are often useful.

The following lemma is evident.

Lemma 2.2. *Let $G = \langle A \rangle$ and $H = \langle B \rangle$ be two groups, $\mathcal{P}(A)$ is the set of palindromes in the alphabets $A^{\pm 1}$, $\mathcal{P}(H)$ is the set of palindromes in the alphabets $B^{\pm 1}$. If $\varphi : G \rightarrow H$ be an epimorphism such that $\varphi(a) = b$ for every $a \in A$ then*

$$pw(H) \leq pw(G).$$

For free nilpotent groups of rank n we have the following set of epimorphisms

$$N_{n,1} \leftarrow N_{n,2} \leftarrow N_{n,3} \leftarrow \dots$$

where

$$N_{n,1} = N_{n,2}/\gamma_2(N_{n,2}), \quad N_{n,2} = N_{n,3}/\gamma_3(N_{n,3}), \dots$$

Applying the above lemma we have:

Corollary 2.1. *The following inequalities hold*

$$(2.2) \quad pw(N_{n,1}) \leq pw(N_{n,2}) \leq pw(N_{n,3}) \leq \dots$$

Lemma 2.3. *Let $G = \langle A \rangle$ be a group generated by a set A . Then the following hold.*

- (1) *If p is a palindrome, then for m in \mathbb{Z} , p^m is also a palindrome.*
- (2) *Any element in G which is conjugate to a product of n palindromes, $n \geq 1$, is a product of n palindromes if n is even, and $n+1$ palindromes if n is odd.*
- (3) *In G any commutator of the type $[u, p]$, where p is a palindrome is a product of 3 palindromes. Any element $[u, a^\alpha]a^\beta$, $a \in A$, $\alpha, \beta \in \mathbb{Z}$, is a product of 3 palindromes.*
- (4) *In G any commutator of the type $[u, pq]$, where p, q are palindromes is a product of 4 palindromes. Any element $[u, pa^\alpha]a^\beta$, $a \in A$, $\alpha, \beta \in \mathbb{Z}$, is a product of 4 palindromes.*

Proof. 1) Let $p = ua^\alpha \bar{u}$, where u is as above. Then $p^2 = ua^\alpha \bar{u}ua^\alpha \bar{u}$, $p^3 = ua^\alpha \bar{u}ua^\alpha \bar{u}ua^\alpha \bar{u}$ are palindromes. The result now follows by induction.

- 2) Let $v = u^{-1}pu$ be a conjugated to the palindrome p . If \bar{u} is the reverse to u , then

$$v = (u^{-1} p \overline{u^{-1}}) \cdot \bar{u} u$$

and we see that $u^{-1} p \overline{u^{-1}}$ and $\bar{u} u$ are palindromes.

If v is the conjugate to the product of $2m$ palindromes p_1, \dots, p_{2m} , for some $u \in G$ let $v = u^{-1}p_1p_2 \dots p_{2m}u$. Then

$$v = (u^{-1} p_1 \overline{u^{-1}})(\bar{u} p_2 u)(u^{-1} p_3 \overline{u^{-1}}) \dots \bar{u} p_{2m} u$$

is the product of $2m$ palindromes.

If $v = u^{-1}p_1p_2 \cdots p_{2m}p_{2m+1}u$, then

$$v = u^{-1}p_1 \cdots p_{2m} \overline{u^{-1}} \overline{u} p_{2m+1}u = u^{-1}p_1 \cdots p_{2m}u \cdot (u^{-1} \overline{u^{-1}}) \cdot \overline{u} p_{2m+1}u.$$

By the previous $u^{-1}p_1 \cdots p_{2m}u$ is a product of $2m$ palindromes, $u^{-1} \overline{u^{-1}}$ and $\overline{u} p_{2m+1}u$ are palindromes. Hence, v is a product of $2m + 2$ palindromes.

3) We can check that

$$[u, p] = u^{-1}p^{-1}up = u^{-1}p^{-1} \overline{u^{-1}} \cdot \overline{u}u \cdot p$$

and $u^{-1}p^{-1} \overline{u^{-1}}$ and $\overline{u}u$ are palindromes. If we take $p = a^\alpha$ then it is clear that $[u, a^\alpha]a^\beta$ is the product of three palindromes.

4) Since $[u, pq] = (u^{-1}(q^{-1}p^{-1})u)pq$ and $u^{-1}(pq)^{-1}u^{-1}$ is a product of two palindromes by (2), hence the result follows. \square

Lemma 2.4. *Let $F_2 = \langle x, y \rangle$ be the free group of rank 2. Then commutator $[y, x]$ is a product of 3 palindromes and any commutator*

$$[[y, x], y], [[y, x], x]$$

is a product of 2 palindromes.

Proof. The first part follows from Lemma 2.3(3).

The second part follows from the formulas

$$[[y, x], y] = [x, y]y^{-1}[y, x]y = (x^{-1}y^{-1}xy^{-1}x^{-1})(yxy),$$

$$\begin{aligned} [[y, x], x] &= [x, y]x^{-1}[y, x]x \\ &= (x^{-1}y^{-1}xy)x^{-1}(y^{-1}x^{-1}yx)x \\ &= (x^{-1}y^{-1}x^{-1})(x^2yx^{-1}y^{-1}x^{-1}yx^2). \end{aligned}$$

\square

Proposition 2.1. *Let G be a group and element g in the center of G is a product of 2 palindromes. Then for any integer m the power g^m is a product of 2 palindromes.*

Proof. At first, let $m > 0$. Use induction on m . If $g = p_1p_2$ is a product of 2 palindromes then

$$g^2 = p_1p_2 \cdot g = p_1gp_2 = p_1^2 \cdot p_2^2$$

and by Lemma 2.3(1) p_1^2 and p_2^2 are palindromes. Assume the result for some m . Then

$$g^{m+1} = (p_1p_2)^m \cdot g = p_1^m gp_2^m = p_1^{m+1} \cdot p_2^{m+1}$$

and by Lemma 2.3(1) p_1^{m+1} and p_2^{m+1} are palindromes.

If $m < 0$ then $g^m = (g^{-m})^{-1}$ and the result follows from the previous case and the fact that inverse to a palindrome is a palindrome. \square

3. PROOF OF THEOREM 1.1

3.1. Palindromic width of r -step free nilpotent groups. Let $N_{n,r} = \langle x_1, x_2, \dots, x_n \rangle$ be a free r -step nilpotent group of rank $n \geq 2$. Let \mathcal{P} be the set of all palindromes in $N_{n,r}$. Note that an element $p \in N_{n,r}$ is a palindrome if it can be presented in the form

$$(3.1) \quad p = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k} x_{i_{k+1}}^{\alpha_{k+1}} x_{i_k}^{\alpha_k} \dots x_{i_2}^{\alpha_2} x_{i_1}^{\alpha_1}$$

where

$$i_j \in \{1, 2, \dots, n\}, \quad \alpha_j \in \mathbb{Z} \setminus \{0\}.$$

Let $N_{n,r}$ is the free r -step nilpotent group with free generators x_1, \dots, x_n . Then

Lemma 3.1. $pw(N_{n,1}) = n$.

Proof. In this case $N_{n,1}$ is a free abelian group of rank n . Since any element $g \in N_{n,1}$ has a form

$$g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad \alpha_i \in \mathbb{Z},$$

then g is a product of n palindromes $x_i^{\alpha_i}$ and hence

$$pw(N_{n,1}) \leq n.$$

To prove the equality we shall show that $lp(x_1 x_2 \dots x_n) = n$. For this define a map

$$- : N_{n,1} \longrightarrow \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_n,$$

where \mathbb{Z}_2 is a cyclic group of order 2 by the rule

$$\overline{g} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n),$$

where

$$\varepsilon_i = \begin{cases} 0 & \text{if } \alpha_i \text{ is even,} \\ 1 & \text{if } \alpha_i \text{ is odd.} \end{cases}$$

Evidently, that for any $g, h \in G$ we have $\overline{gh} = \overline{g} + \overline{h}$.

If a palindrome p has a form (3.1) then

$$\overline{p} = (\nu_1, \nu_2, \dots, \nu_n)$$

contains no more than one non-zero component. On the other hand

$$\overline{x_1 x_2 \dots x_n} = (1, 1, \dots, 1).$$

Then $x_1 x_2 \dots x_n$ is a product of at least n palindromes. □

Lemma 3.2. For $r \geq 2$, $n \leq pw(N_{n,r}) \leq 3n$.

Proof. At first, we claim that any element g in $N_{n,r}$ can be represented in the form

$$g = [u_1, x_1] x_1^{\alpha_1} [u_2, x_2] x_2^{\alpha_2} \dots [u_n, x_n] x_n^{\alpha_n}.$$

Indeed, use induction on the step of nilpotency r . If $r = 2$ and $g \in N_{n,2}$ then by Lemma 2.1,

$$g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} [u_1, x_1] [u_2, x_2] \dots [u_n, x_n], \quad \alpha_i \in \mathbb{Z}, \quad u_i \in N_{n,2}.$$

But the commutators $[u_i, x_i]$, $i = 1, 2, \dots, n$, lie in the center of $N_{n,2}$. Hence

$$g = [u_1, x_1] x_1^{\alpha_1} [u_2, x_2] x_2^{\alpha_2} \dots [u_n, x_n] x_n^{\alpha_n}.$$

has the required form. Assume the result for groups $N_{n,r}$ and consider $N_{n,r+1}$. Let $\Gamma = \gamma_{r+1}(N_{n,r+1}) = [\gamma_r(N_{n,r+1}), N_{n,r+1}]$. Then an element g of $N_{n,r+1}$ has the form

$$g = [u_1, x_1] x_1^{\alpha_1} [u_2, x_2] x_2^{\alpha_2} \dots [u_n, x_n] x_n^{\alpha_n} d$$

for some $d \in \Gamma$. By [2, Lemma 3],

$$d = [a_1, x_1] [a_2, x_2] \dots [a_n, x_n], \text{ for some } a_i \in \Gamma.$$

Since all $[a_i, x_i]$ lie in the center of $N_{n,r+1}$ then

$$\begin{aligned} g &= [u_1, x_1] [a_1, x_1] x_1^{\alpha_1} [u_2, x_2] [a_2, x_2] x_2^{\alpha_2} \dots [u_n, x_n] [a_n, x_n] x_n^{\alpha_n} = \\ &= [u_1 a_1, x_1] x_1^{\alpha_1} [u_2 a_2, x_2] x_2^{\alpha_2} \dots [u_n a_n, x_n] x_n^{\alpha_n} \end{aligned}$$

has the required form.

By Lemma 2.3(3) any element $[u_i, x_i] x_i^{\alpha_i}$ is a product of 3 palindromes and g is a product of $3n$ palindromes. The lower bound follows from Lemma 3.1 and Corollary 2.1. \square

3.2. Palindromic width of 2-step free nilpotent groups. In the following we investigate a few special cases where we improve the bound of $pw(N_{n,r})$.

3.3. $r = 2$. In this subsection we will consider 2-step nilpotent groups $N_{n,2}$. We know that any palindrome has the form $p = u x_l^\beta \bar{u}$, where

$$u = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k}$$

and

$$\bar{u} = x_{i_k}^{\alpha_k} x_{i_{k-1}}^{\alpha_{k-1}} \dots x_{i_1}^{\alpha_1}$$

is its reverse. Prove that we can assume that

$$i_1 < i_2 < \dots < i_k$$

and

$$l \notin \{i_1, i_2, \dots, i_k\}.$$

Indeed, if

$$p = u_1 x_i^{\alpha_i} x_j^{\alpha_j} p_0 x_j^{\alpha_j} x_i^{\alpha_i} \bar{u}_1$$

for some palindrome p_0 then

$$p = u_1 x_j^{\alpha_j} x_i^{\alpha_i} [x_i, x_j]^{\alpha_i \alpha_j} p_0 x_i^{\alpha_i} x_j^{\alpha_j} [x_j, x_i]^{\alpha_i \alpha_j} \bar{u}_1 = u_1 x_j^{\alpha_j} x_i^{\alpha_i} p_0 x_i^{\alpha_i} x_j^{\alpha_j} \bar{u}_1.$$

Hence, we can permute any elements in u and the element p does not change.

Let $N_{2,2} = \langle x, y \rangle$ be the free nilpotent group of rank 2. Any element in this group has a presentation

$$x^\alpha y^\beta [y, x]^\gamma, \quad \alpha, \beta, \gamma \in \mathbb{Z}.$$

Lemma 3.3. *For some integers a and b , any palindrome in $N_{2,2}$ has one of the following form:*

$$p_{(2a,b)} = x^{2a} y^b z^{ab}, \quad p_{(a,2b)} = x^a y^{2b} z^{ab}, \quad \text{where } z = [y, x].$$

Proof. Let p be a some palindrome in $N_{2,2}$. Induction by the syllable length of p . If it is equal to 1 then $p = x^a$ or $p = y^b$. If the syllable length is 3 then,

$$p = x^\alpha y^\beta x^\alpha = x^{2\alpha} y^\beta [y, x]^{\alpha\beta}$$

or

$$p = y^\alpha x^\beta y^\alpha = x^\beta y^{2\alpha} [y, x]^{\alpha\beta}.$$

Using the note before the lemma, we see that all other possibilities are reduced to these two cases. □

We see that if palindrome lies in the commutator subgroup $N'_{2,2}$ then it is trivial. More generally, we have the following.

Lemma 3.4. *If a product of two palindromes lies in $N'_{2,2}$ then this product is trivial.*

Proof. We know that any palindrome has the form $p_{(2a,b)}$ or $p_{(a,2b)}$. Consider the product of two palindromes. We have to check four possibilities: both palindromes have type $p_{(2a,b)}$ or $p_{(a,2b)}$; one palindrome has type $p_{(2a,b)}$ and another has the type $p_{(a,2b)}$. If both palindromes have type the $p_{(2a,b)}$ then their product

$$p_{(2a_1,b_1)} \cdot p_{(2a_2,b_2)} = x^{2a_1} y^{b_1} z^{a_1 b_1} \cdot x^{2a_2} y^{b_2} z^{a_2 b_2} = x^{2(a_1+a_2)} y^{b_1+b_2} z^{b_1(a_1+2a_2)+a_2 b_2}$$

lies in the commutator subgroup if and only if

$$\begin{cases} a_1 + a_2 = 0, \\ b_1 + b_2 = 0, \end{cases}$$

or

$$\begin{cases} a_1 = -a_2, \\ b_1 = -b_2. \end{cases}$$

But this means that

$$p_{(2a_1,b_1)} \cdot p_{(2a_2,b_2)} = z^{-b_2 a_2 + a_2 b_2} = z^0 = 1.$$

The case of a product $p_{(a_1,2b_1)} \cdot p_{(a_2,2b_2)}$ is similar.

Consider a product of palindromes of different types:

$$p_{(2a_1,b_1)} \cdot p_{(a_2,2b_2)} = x^{2a_1} y^{b_1} z^{a_1 b_1} \cdot x^{a_2} y^{2b_2} z^{a_2 b_2} = x^{2a_1+a_2} y^{b_1+2b_2} z^{a_1 b_1 + a_2 b_2 + b_1 a_2}.$$

We see that this product lies in the commutator subgroup if and only if

$$\begin{cases} 2a_1 + a_2 = 0, \\ b_1 + 2b_2 = 0, \end{cases}$$

or

$$\begin{cases} a_2 = -2a_1, \\ b_1 = -2b_2. \end{cases}$$

But this means that

$$p_{(2a_1,b_1)} \cdot p_{(a_2,2b_2)} = z^{b_1(a_1+a_2)+a_2 b_2} = z^0 = 1.$$

The case of the product $p_{(a_2,2b_2)} \cdot p_{(2a_1,b_1)}$ is similar. □

Proposition 3.1. $pw(N_{2,2}) = 3$.

Proof. At first, we prove that any element in $N_{2,2}$ is a product of 3 palindromes. Note that $[y, x]$ is an element in the center of G . Note that

$$\begin{aligned} x^\alpha y^\beta [y, x]^\gamma &= x^\alpha y^{\beta-\gamma} y^\gamma [y, x]^\gamma \\ &= x^\alpha y^{\beta-\gamma} (x^{-1} y x)^\gamma. \end{aligned}$$

It follows that

$$x^\alpha y^\beta z^\gamma = x^\alpha y^{\beta-\gamma} y^\gamma x^\alpha \cdot x^{-\alpha-2} \cdot x y^\gamma x.$$

Hence, $pw(N_{2,2}) \leq 3$. On the other hand we proved in Lemma 3.4 that z is not a product of 2 palindromes. Hence $pw(N_{2,2}) \geq 3$. \square

In the general case we can prove

Proposition 3.2. *Any element in $N_{n,2}$, $n \geq 2$ is a product of $3(n-1)$ palindromes.*

Proof. Let $g \in N_{n,2}$. Then g has the form

$$g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \prod_{1 \leq j < i \leq n} [x_i, x_j]^{\gamma_{ij}}$$

for some integers α_i and γ_{ij} . Using the commutator identities (cf. for eg. [17]) we have

$$\prod_{1 \leq j < i \leq n} [x_i, x_j]^{\gamma_{ij}} = [x_n^{\gamma_{n1}} x_{n-1}^{\gamma_{n-1,1}} \dots x_2^{\gamma_{21}}, x_1] [x_n^{\gamma_{n2}} x_{n-1}^{\gamma_{n-1,2}} \dots x_3^{\gamma_{32}}, x_2] \dots$$

$$[x_n^{\gamma_{n,n-2}} x_{n-1}^{\gamma_{n-1,n-2}}, x_{n-2}] [x_n^{\gamma_{n,n-1}}, x_{n-1}]$$

Since, the commutator subgroup $N'_{n,2}$ is equal to the center of $N_{n,2}$ then

$$\begin{aligned} g &= [x_n^{\gamma_{n1}} x_{n-1}^{\gamma_{n-1,1}} \dots x_2^{\gamma_{21}}, x_1] x_1^{\alpha_1} \cdot [x_n^{\gamma_{n2}} x_{n-1}^{\gamma_{n-1,2}} \dots x_3^{\gamma_{32}}, x_2] x_2^{\alpha_2} \cdot \dots \cdot \\ &\quad [x_n^{\gamma_{n,n-2}} x_{n-1}^{\gamma_{n-1,n-2}}, x_{n-2}] x_{n-2}^{\alpha_{n-2}} \cdot x_{n-1}^{\alpha_{n-1}} [x_n^{\gamma_{n,n-1}}, x_{n-1}] x_n^{\alpha_n}. \end{aligned}$$

By Lemma 2.3(2) any element

$$[x_n^{\gamma_{n1}} x_{n-1}^{\gamma_{n-1,1}} \dots x_2^{\gamma_{21}}, x_1] x_1^{\alpha_1}, [x_n^{\gamma_{n2}} x_{n-1}^{\gamma_{n-1,2}} \dots x_3^{\gamma_{32}}, x_2] x_2^{\alpha_2}, \dots, [x_n^{\gamma_{n,n-2}} x_{n-1}^{\gamma_{n-1,n-2}}, x_{n-2}] x_{n-2}^{\alpha_{n-2}}$$

is a product of 3 palindromes. Elements x_{n-1} and x_n generate a group which is isomorphic to $N_{2,2}$ and by Proposition 3.1, the element $x_{n-1}^{\alpha_{n-1}} [x_n^{\gamma_{n,n-1}}, x_{n-1}] x_n^{\alpha_n}$ is a product of 3 palindromes. Hence, g is a product of $3(n-1)$ palindromes. \square

3.3.1. *Palindromic width in $N_{3,2}$.* Any element in $N_{3,2}$ has a form

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} [x_2, x_1]^{\gamma_{21}} [x_3, x_1]^{\gamma_{31}} [x_3, x_2]^{\gamma_{32}} \text{ for some } \alpha_i, \gamma_{ij} \in \mathbb{Z}.$$

We will denote the basis commutators by

$$z_{ij} = [x_i, x_j], \quad 1 \leq j < i \leq n.$$

As in the case $n = 2$ we can prove

Lemma 3.5. *There are three different types of palindromes in $N_{3,2}$:*

$$\begin{aligned} p_{(\alpha_0, 2\alpha_1, 2\alpha_2)} &= x_1^{\alpha_0} x_2^{2\alpha_1} x_3^{2\alpha_2} z_{32}^{2\alpha_1\alpha_2} z_{31}^{\alpha_0\alpha_2} z_{21}^{\alpha_0\alpha_1}, \\ p_{(2\alpha_1, \alpha_0, 2\alpha_2)} &= x_1^{2\alpha_1} x_2^{\alpha_0} x_3^{2\alpha_2} z_{32}^{\alpha_0\alpha_2} z_{31}^{2\alpha_1\alpha_2} z_{21}^{\alpha_0\alpha_1}, \\ p_{(2\alpha_1, 2\alpha_2, \alpha_0)} &= x_1^{2\alpha_1} x_2^{2\alpha_2} x_3^{\alpha_0} z_{32}^{\alpha_0\alpha_2} z_{31}^{\alpha_0\alpha_1} z_{21}^{2\alpha_1\alpha_2}. \end{aligned}$$

Lemma 3.6. *Any element in $N'_{3,2}$ is a product of 4 palindromes.*

Proof. At first we prove that any element in the commutator subgroup $N'_{3,2}$ is a commutator of the form

$$[x_1^a x_2^b x_3^c, x_1^k x_2^l] \text{ for some integers } a, b, c, k, l.$$

We follow the ideas of Allambergenov-Roman'kov [4]. Any element in $N'_{3,2}$ has a form

$$[x_3, x_2]^{\gamma_{32}} [x_3, x_1]^{\gamma_{31}} [x_2, x_1]^{\gamma_{21}} \text{ for some integers } \gamma_{ij}.$$

Represent the commutator $[x_1^a x_2^b x_3^c, x_1^k x_2^l]$ as a product of basis commutators

$$[x_1^a x_2^b x_3^c, x_1^k x_2^l] = [x_3, x_2]^{cl} [x_3, x_1]^{ck} [x_2, x_1]^{bk-al}.$$

To prove the assertion we have to prove that the following system

$$(3.2) \quad \begin{cases} cl = \gamma_{32}, \\ ck = \gamma_{31}, \\ bk - al = \gamma_{21}, \end{cases}$$

with the variables a, b, c, k, l has an integer solution for any integers $\gamma_{32}, \gamma_{31}, \gamma_{21}$. Let $d = (\gamma_{32}, \gamma_{31})$ be the greatest common divisor of γ_{32} and γ_{31} . Then $\gamma_{32} = d\gamma'_{32}$, $\gamma_{31} = d\gamma'_{31}$ for some integers $\gamma'_{32}, \gamma'_{31}$ and $(\gamma'_{32}, \gamma'_{31}) = 1$. Take

$$c = d, \quad l = \gamma'_{32}, \quad k = \gamma'_{31}.$$

Since $(k, l) = 1$, the last equation also is decidable. Hence, any element in $N'_{3,2}$ is a commutator. Now the assertion follows from Lemma 2.3(4). \square

3.4. Palindromic width in $\overline{N}_{3,2}$. Now consider the group $\overline{N}_{n,2} = N_{n,2}/\langle x_1^2, \dots, x_n^2 \rangle$. For simplicity of notation, we shall often forget the 'bar' from \bar{x}_i, \bar{z}_{ij} etc in $\overline{N}_{3,2}$ and shall continue denoting then as x_i, z_{ij} etc unless specified otherwise.

When $n = 3$, then it follows from Lemma 3.5 that palindromes in $\overline{N}_{n,2}$ are of the following form:

$$(3.3) \quad \bar{p}_{(\alpha_0, 2\alpha_1, 2\alpha_2)} = x_1^{\alpha_0} z_{31}^{\alpha_0\alpha_2} z_{21}^{\alpha_0\alpha_1},$$

$$(3.4) \quad \bar{p}_{(2\alpha_1, \alpha_0, 2\alpha_2)} = x_2^{\alpha_0} z_{32}^{\alpha_0\alpha_2} z_{21}^{\alpha_0\alpha_1},$$

$$(3.5) \quad \bar{p}_{(2\alpha_1, 2\alpha_2, \alpha_0)} = x_3^{\alpha_0} z_{32}^{\alpha_0\alpha_2} z_{31}^{\alpha_0\alpha_1}.$$

Further, observe that there is an onto map $\pi : \mathcal{P}(N_{3,2}) \rightarrow \mathcal{P}(\overline{N}_{3,2})$.

In the following, for simplicity, we denote the palindromes of the form (3.3), (3.4) and (3.5) by \bar{p}_1, \bar{p}_2 and \bar{p}_3 respectively forgetting the subscript. When we write a product, for eg. $\bar{p}_1 \bar{p}_1 \bar{p}_1$, it should be understood that each \bar{p}_1 is a palindrome of the type (3.3) but not necessarily with the same subscript unless it is mentioned otherwise.

Theorem 3.1. $pw(\overline{N}_{3,2}) = 4$.

As an immediate corollary to the theorem we obtain

Corollary 3.1. $4 \leq pw(N_{3,2}) \leq 6$.

Proof. Since the projection map $N_{3,2} \rightarrow \overline{N}_{3,2}$ is onto, it follows from Lemma 2.2 and the above proposition that $pw(N_{3,2}) \geq 4$.

If we take arbitrary element $g \in N_{3,2}$ then it follows from Lemma 3.6 that we can represent it in the form

$$g = x_1^{\alpha_1} \cdot [x_1^a x_2^b x_3^c, x_1^k x_2^l] x_2^{\alpha_2} \cdot x_3^{\alpha_3}.$$

In view of Lemma 2.3(3) the element $[x_1^a x_2^b x_3^c, x_1^k x_2^l] x_2^{\alpha_2}$ is a product of 4 palindromes. Hence, g is a product of 6 palindromes. \square

Now we shall start proving Theorem 3.1. First we prove the following lemma.

Lemma 3.7. *For $i, j \in \{1, 2, 3\}$, let $z_{ij} = [x_i, x_j]$ in $\overline{N}_{3,2}$. The element $g = z_{21} z_{31} z_{32}$ in $\overline{N}_{3,2}$ has palindromic length is at least 4.*

Proof. If possible, suppose $g = \overline{p}_i \overline{p}_j \overline{p}_k$ is a product of three palindromes, where $i, j, k \in \{1, 2, 3\}$.

Case (i): $i = j = k = 1$. Note that after simplifying we have

$$(3.6) \quad \overline{p}_1 \overline{p}_1 \overline{p}_1 = x_1^{\gamma_1} z_{31}^{\alpha_1 \gamma_1} z_{21}^{\beta_1 \gamma_1} x_1^{\gamma_2} z_{31}^{\alpha_2 \gamma_2} z_{21}^{\beta_2 \gamma_2} x_1^{\gamma_3} z_{31}^{\alpha_3 \gamma_3} z_{21}^{\beta_3 \gamma_3}$$

$$(3.7) \quad = x_1^{\gamma_1 + \gamma_2 + \gamma_3} z_{21}^{\beta_1 \gamma_1 + \beta_2 \gamma_2 + \beta_3 \gamma_3} z_{31}^{\alpha_1 \gamma_1 + \alpha_2 \gamma_2 + \alpha_3 \gamma_3}.$$

The product on the RHS does not contain z_{32} . Hence this product can not be equal to g , i.e. $g \neq \overline{p}_1 \overline{p}_1 \overline{p}_1$.

Similarly, we see that $g \neq \overline{p}_i \overline{p}_i \overline{p}_i$ for $i = 2, 3$.

Case (ii): All indices i, j, k are different. Then there are six such choices. Suppose

$$h = x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} c, \quad c \in \overline{N}'_{3,2}.$$

If $h = g$ then $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and hence $h = e$, the identity element. Thus g can not be equal to h .

Similarly $g \neq \overline{p}_i \overline{p}_j \overline{p}_k$ for mutually distinct i, j, k .

Case (iii): Suppose in the set $\{i, j, k\}$ two elements are equal. For example, if two of them are equal to 1 and the other 2, then we have the following cases:

$$\overline{p}_2 \overline{p}_1 \overline{p}_1, \overline{p}_1 \overline{p}_2 \overline{p}_1, \overline{p}_1 \overline{p}_1 \overline{p}_2.$$

We have

$$\begin{aligned} \overline{p}_2 \overline{p}_1 \overline{p}_1 &= x_2^{\gamma_1} z_{32}^{\alpha_2 \gamma_2} z_{21}^{\beta_2 \gamma_2} x_1^{\gamma_1} z_{31}^{\alpha_1 \gamma_1} z_{21}^{\beta_1 \gamma_1} x_1^{\gamma_1} z_{31}^{\alpha_1 \gamma_1} z_{21}^{\beta_1 \gamma_1} \\ &= x_2^{\gamma_1} z_{32}^{\alpha_2 \gamma_2} z_{21}^{\beta_2 \gamma_2} x_1^{\gamma_2 + \gamma_3} z_{21}^{\beta_2 \gamma_2 + \beta_3 \gamma_3} z_{31}^{\alpha_2 \gamma_2 + \alpha_3 \gamma_3} \\ &= x_1^{\gamma_2 + \gamma_3} x_2^{\gamma_1} z_{21}^{\beta_1 \gamma_1 + \beta_2 \gamma_2 + \beta_3 \gamma_3 + \gamma_1(\gamma_2 + \gamma_3)} z_{31}^{\alpha_2 \gamma_2 + \alpha_3 \gamma_3} z_{32}^{\alpha_1 \gamma_1}. \end{aligned}$$

If $g = \overline{p}_2 \overline{p}_1 \overline{p}_1$ then $\gamma_1 = 0$, $\gamma_2 + \gamma_3 = 0$; this implies $\overline{p}_2 \overline{p}_1 \overline{p}_1 = e$, the identity element, which is a contradiction.

Next consider $\overline{p}_1 \overline{p}_2 \overline{p}_1$. Observe that

$$\overline{p}_1 \overline{p}_2 \overline{p}_1 = x_1^{\gamma_1 + \gamma_3} x_2^{\gamma_2} z_{21}^{\beta_1 \gamma_1 + \beta_2 \gamma_2 + \gamma_3(\beta_3 + \gamma_2)} z_{31}^{\alpha_1 \gamma_1 + \alpha_3 \gamma_3} z_{32}^{\alpha_2 \gamma_2}.$$

If this product is equal to g then $\gamma_2 = 0$ and $\gamma_1 = -\gamma_3$ which implies that the product equals the identity element. Hence g can not be equal to $\overline{p}_1 \overline{p}_2 \overline{p}_1$. Similarly, g can not be equal to $\overline{p}_1 \overline{p}_1 \overline{p}_2$.

Suppose in the set $\{i, j, k\}$ two elements are equal to 1 and the third is equal to 3 then, we have the following cases:

$$\bar{p}_3\bar{p}_1\bar{p}_1, \bar{p}_1\bar{p}_3\bar{p}_1, \bar{p}_1\bar{p}_1\bar{p}_3.$$

Note that

$$\begin{aligned} \bar{p}_3\bar{p}_1\bar{p}_1 &= x_3^{\gamma_1} z_{32}^{\alpha_1\gamma_1} z_{31}^{\beta_1\gamma_1} x_1^{\gamma_2+\gamma_3} z_{21}^{\beta_2\gamma_2+\beta_3\gamma_3} z_{31}^{\alpha_2\gamma_2+\alpha_3\gamma_3} \\ &= x_1^{\gamma_2+\gamma_3} x_3^{\gamma_1} z_{21}^{\beta_2\gamma_2+\beta_3\gamma_3} z_{31}^{\alpha_2\gamma_2+\alpha_3\gamma_3+\gamma_1(\gamma_2+\gamma_3)} z_{32}^{\alpha_1\gamma_1}. \end{aligned}$$

If $g = \bar{p}_3\bar{p}_1\bar{p}_1$ then, $\gamma_2 + \gamma_3 = \gamma_1 = 0$ and this implies $e = \bar{p}_3\bar{p}_1\bar{p}_1$. Thus $g \neq \bar{p}_3\bar{p}_1\bar{p}_1$.

If $g = \bar{p}_1\bar{p}_1\bar{p}_3$ then

$$\begin{aligned} \bar{p}_1\bar{p}_1\bar{p}_3 &= x_1^{\gamma_1} z_{31}^{\alpha_1\gamma_1} z_{21}^{\beta_1\gamma_1} x_1^{\gamma_2} z_{31}^{\alpha_2\gamma_2} z_{21}^{\beta_2\gamma_2} x_3^{\gamma_3} z_{32}^{\alpha_3\gamma_3} z_{31}^{\beta_3\gamma_3} \\ &= x_1^{\gamma_1+\gamma_3} x_3^{\gamma_3} z_{21}^{\beta_1\gamma_1+\beta_2\gamma_2} z_{31}^{\alpha_1\gamma_1+\alpha_2\gamma_2+\beta_3\gamma_3} z_{32}^{\alpha_3\gamma_3}. \end{aligned}$$

If $g = \bar{p}_1\bar{p}_1\bar{p}_3$ then

$$\gamma_1 + \gamma_2 = 0, \gamma_3 = 0, \beta_1\gamma_1 + \beta_2\gamma_2 = 1, \alpha_1\gamma_1 + \alpha_2\gamma_2 + \beta_3\gamma_3 = 1, \alpha_3\gamma_3 = 1.$$

But the second and the last equality can not hold together. Thus $g \neq \bar{p}_1\bar{p}_1\bar{p}_3$.

Similarly, we can see that $g \neq \bar{p}_1\bar{p}_3\bar{p}_1$.

Suppose two indices are 2 and the other 1. We see that

$$\begin{aligned} \bar{p}_2\bar{p}_2\bar{p}_1 &= x_2^{\gamma_1} z_{32}^{\alpha_1\gamma_1} z_{21}^{\beta_1\gamma_1} x_2^{\gamma_2} z_{32}^{\alpha_2\gamma_2} z_{21}^{\beta_2\gamma_2} x_1^{\gamma_3} z_{31}^{\alpha_3\gamma_3} z_{21}^{\beta_3\gamma_3} \\ &= x_1^{\gamma_3} x_2^{\gamma_1+\gamma_2} z_{21}^{\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3+\gamma_3(\gamma_1+\gamma_2)} z_{31}^{\alpha_3\gamma_3} z_{32}^{\alpha_1\gamma_1+\alpha_2\gamma_2}. \end{aligned}$$

If $g = \bar{p}_2\bar{p}_2\bar{p}_1$ then the following systems hold:

$$\begin{aligned} \gamma_3 = 0, \gamma_1 + \gamma_2 = 0, \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 + \gamma_3(\gamma_1 + \gamma_2) &= 1, \\ \alpha_3\gamma_3 = 1, \alpha_1\gamma_1 + \alpha_2\gamma_2 &= 1. \end{aligned}$$

We see that the first and fourth equation can not hold simultaneously. So g can not be equal to $\bar{p}_2\bar{p}_2\bar{p}_1$. Similarly, $g \neq \bar{p}_2\bar{p}_1\bar{p}_2$ and $g \neq \bar{p}_1\bar{p}_2\bar{p}_2$.

Next suppose two indices are 2 and the other 3. Let $g = \bar{p}_2\bar{p}_2\bar{p}_3$. Note that

$$\begin{aligned} \bar{p}_2\bar{p}_2\bar{p}_3 &= x_2^{\gamma_1} z_{32}^{\alpha_1\gamma_1} z_{21}^{\beta_1\gamma_1} x_2^{\gamma_2} z_{32}^{\alpha_2\gamma_2} z_{21}^{\beta_2\gamma_2} x_3^{\gamma_3} z_{32}^{\alpha_3\gamma_3} z_{31}^{\beta_3\gamma_3} \\ &= x_2^{\gamma_1+\gamma_2} x_3^{\gamma_3} z_{21}^{\beta_1\gamma_1+\beta_2\gamma_2} z_{31}^{\beta_3\gamma_3} z_{32}^{\alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3}. \end{aligned}$$

If $g = \bar{p}_2\bar{p}_2\bar{p}_3$ then we see that $\gamma_3 = 0$ and $\beta_3\gamma_3 = 1$ which is a contradiction. Hence $g \neq \bar{p}_2\bar{p}_2\bar{p}_3$. Similarly, $g \neq \bar{p}_2\bar{p}_3\bar{p}_2$ and $g \neq \bar{p}_3\bar{p}_2\bar{p}_2$.

Suppose two indices are 3 and the other 1. Observe that

$$\begin{aligned} \bar{p}_1\bar{p}_3\bar{p}_3 &= x_1^{\gamma_1} z_{31}^{\alpha_1\gamma_1} z_{21}^{\beta_1\gamma_1} x_3^{\gamma_2} z_{32}^{\alpha_2\gamma_2} z_{31}^{\beta_2\gamma_2} x_3^{\gamma_3} z_{32}^{\alpha_3\gamma_3} z_{31}^{\beta_3\gamma_3} \\ &= x_1^{\gamma_1} x_3^{\gamma_2+\gamma_3} z_{21}^{\beta_1\gamma_1} z_{31}^{\alpha_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3} z_{32}^{\alpha_2\gamma_2+\alpha_3\gamma_3}. \end{aligned}$$

If $g = \bar{p}_1\bar{p}_3\bar{p}_3$, we see that $\gamma_1 = 0$ and $\beta_1\gamma_1 = 1$ which is a contradiction. Thus g can not be equal to $\bar{p}_1\bar{p}_3\bar{p}_3$. Similarly $g \neq \bar{p}_3\bar{p}_3\bar{p}_1$ and $g \neq \bar{p}_3\bar{p}_1\bar{p}_3$.

Suppose two indices are 3 and other 2. Note that

$$\begin{aligned} \bar{p}_3\bar{p}_2\bar{p}_3 &= x_3^{\gamma_1} z_{32}^{\alpha_1\gamma_1} z_{31}^{\beta_1\gamma_1} x_2^{\gamma_2} z_{32}^{\alpha_2\gamma_2} z_{21}^{\beta_2\gamma_2} x_3^{\gamma_3} z_{32}^{\alpha_3\gamma_3} z_{31}^{\beta_3\gamma_3} \\ &= x_2^{\gamma_2} x_3^{\gamma_1+\gamma_3} z_{\beta_2\gamma_2} z_{31}^{\beta_1\gamma_1+\beta_3\gamma_3} z_{32}^{\alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3+\gamma_1\gamma_2}. \end{aligned}$$

If $g = \bar{p}_3\bar{p}_2\bar{p}_3$ then $\gamma_2 = 0$ and $\beta_2\gamma_2 = 1$, contradiction. So g can not be equal to $\bar{p}_3\bar{p}_2\bar{p}_3$. Similarly, $g \neq \bar{p}_3\bar{p}_3\bar{p}_2$ and $g \neq \bar{p}_2\bar{p}_3\bar{p}_3$.

Thus we see that $g = z_{21}z_{31}z_{32}$ can not be expressed as a product of three palindromes. On the other hand, note that

$$\begin{aligned} g &= [x_2, x_1][x_3, x_1][x_3, x_2] \\ &= [x_3, x_2][x_2, x_1][x_3, x_1] \\ &= x_2[x_3, x_2].x_2[x_2, x_1].x_1[x_3, x_1].x_1 \\ &= \bar{p}_2\bar{p}_2\bar{p}_1\bar{p}_1 \end{aligned}$$

Thus g can be expressed as a product of four palindromes. \square

Corollary 3.2. *The element $g = x_1^2x_2^2x_3^2z_{21}z_{31}z_{32}$ can not be expressed as a product of 3 palindromes in $N_{3,2}$.*

Proof. If possible suppose, $g = p_ip_jp_k$ for $i, j, k \in \{1, 2, 3\}$. Then in $\bar{N}_{3,2}$, $\bar{g} = z_{21}z_{31}z_{32} = \bar{p}_i\bar{p}_j\bar{p}_k$, which is a contradiction due to the above proposition. Hence g can not be written as product of three palindromes. \square

Proof of Theorem 3.1.

Proof. Since there exists a homomorphism $\bar{N}_{3,2} \rightarrow \bar{N}_{3,1}$ then $pw(\bar{N}_{3,2}) \geq 3$. Note that any element g of $\bar{N}_{3,2}$ of the form

$$g = x_1^{a_1} x_2^{a_2} x_3^{a_3} z_{21}^{b_1} z_{31}^{b_2} z_{32}^{b_3}$$

where, for $i = 1, 2, 3$, $a_i, b_i \in \{0, 1\}$. Define

$$|g| = \sum_{i=1}^3 (a_i + b_i).$$

If $|g| = 1$ then $l_{\mathcal{P}}(g) \leq 3$, since, any commutator z_{ij} is a product of two palindromes.

Let $|g| = 2$, then we have 15 possibilities for $(a_1, a_2, a_3, b_1, b_2, b_3)$, where each of the a_i and b_i is either 0 or 1. For simplicity of notation we identify the 6-tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ with the binary word $a_1a_2a_3b_1b_2b_3$ and write down the 15 possibilities below:

110000, 101000, 100100, 100010, 100001, 011000, 010100, 010010, 010001, 001100, 001010, 001001, 000110, 000101, 000011.

In the first twelve cases we have a product of two generators or a product of one generator and a commutator. The palindromic length of this product is ≤ 3 . In the last three cases we have:

$$000110 : g = z_{21}z_{31} = x_2x_1x_2.x_3x_1x_3x_1.$$

$$000101 : g = z_{21}z_{32} = z_{32}z_{21} = x_3x_2x_3.x_1x_2x_1.$$

$$000011 : g = z_{31}z_{32} = x_3x_1x_3x_1x_3.x_2x_3x_2.$$

Thus in each cases g is a product of at most three palindromes.

Let $|g| = 3$, then we have $\binom{6}{3} = 20$ possibilities:

111000, 110100, 110010, 110001, 101100, 101010, 101001, 100110, 100101, 100011, 011100, 011010, 011001, 010110, 010101, 010011, 001110, 001101, 001011, 000111.

After rearranging terms and simplification we get:

$$\begin{aligned}
110010 : g &= x_1x_2z_{31} = z_{31}x_1x_2 = x_3x_2x_3.x_2. \\
110001 : g &= x_1x_2z_{32} = x_1z_{32}x_2 = x_1.x_3x_2x_3; \quad 101100 : g = z_{21}x_1x_3 = x_2x_1x_2.x_3. \\
101010 : g &= x_1x_3z_{31} = x_3x_1; \quad 101001 : g = x_1x_3z_{32} = x_1.x_2x_3x_2. \\
100110 : g &= x_1z_{21}z_{31} = z_{21}x_1z_{31} = x_2x_1x_2x_1.x_1.x_3x_1x_3x_1 = x_2x_1x_2.x_3x_1x_3.x_1. \\
100101 : g &= x_1z_{21}z_{32} = z_{32}z_{21}x_1 = x_3x_2x_3.x_1.x_2. \\
100011 : g &= x_1z_{31}z_{32} = z_{31}x_1z_{32} = x_3x_1.x_2x_3x_2. \\
010110 : g &= x_2z_{21}z_{31} = x_1x_2x_1.x_3x_1x_3.x_1. \quad 010101 : g = x_2z_{21}z_{32} = x_1x_2x_1.x_3x_2x_3.x_2. \\
010011 : g &= z_{31}z_{32}x_2 = x_3x_1x_3.x_1.x_3x_2x_3. \quad 001110 : g = z_{21}x_3z_{31} = x_2x_1x_2.x_3x_1. \\
001101 : g &= z_{21}x_3z_{32} = x_2x_1x_2.x_1.x_2x_3x_2. \quad 001011 : g = x_3z_{31}z_{32} = x_1x_3x_1.x_3x_2x_3.x_2.
\end{aligned}$$

Thus we see that in each of the above cases, g is a product of at most three palindromes. Finally $000111 : g = z_{21}z_{31}z_{32}$ is a product of four palindromes as we have seen in Lemma 3.7.

Let $|g| = 4$. Then we have $\binom{6}{4} = 15$ possibilities:

111100, 111010, 110110, 101110, 011110, 111001, 110101, 101101, 011101, 110011, 101011, 011011, 100111, 010111, 001111.

We have after rearranging terms and simplification,

$$\begin{aligned}
111100 : g &= x_1x_2x_3z_{21} = x_1x_2z_{21}x_3 = x_1x_2x_2x_1x_2x_1x_3 = x_2x_1x_3. \\
110110 : g &= x_1z_{21}z_{31} = z_{31}x_1x_2z_{21} = x_3x_1x_3.x_1x_2x_1. \\
101110 : g &= x_1x_3z_{21}z_{31} = z_{21}x_1x_3z_{31} = x_2x_1x_2.x_1x_3x_1. \\
011110 : g &= x_2z_{21}x_3z_{31} = x_1x_2x_3x_1; \quad 111001 : g = x_1x_2x_3z_{32} = x_1x_3x_2. \\
110101 : g &= x_1x_2z_{21}z_{32} = x_2x_1.x_3x_2x_3x_2 = x_2x_1x_2.x_2x_3x_2x_3x_2. \\
101101 : g &= z_{21}x_1x_3z_{32} = x_2x_1x_2.x_2x_3x_2. \\
011101 : g &= x_2z_{21}x_3z_{32} = x_1x_2x_1.x_2x_3x_2; \quad 110011 : g = z_{31}x_1z_{32}x_2 = x_3x_1x_2x_3. \\
101011 : g &= z_{31}x_1x_3z_{32} = x_3x_1x_3.x_2x_3x_2; \\
011011 : g &= z_{31}x_1z_{32}z_{21} = x_3x_1x_3.x_3x_2x_3.x_1x_2x_1. \\
010111 : g &= x_2z_{21}z_{31}z_{32} = x_1x_2x_1.x_3x_1x_3x_1x_3.x_2x_3x_2. \\
001111 : g &= x_3z_{21}z_{31}z_{32} = z_{21}x_3z_{31}z_{32} = x_2x_1x_2.x_3x_1x_3.x_2x_3x_2.
\end{aligned}$$

Thus we see that in each of the above cases g is a product of at most three palindromes.

Let $|g| = 5$. There are six possibilities and after rearranging terms and simplification we have:

$$\begin{aligned}
111110 : g &= x_1x_2x_3z_{21}z_{31} = x_1x_2z_{21}x_3z_{31} = x_2x_3x_1. \\
111101 : g &= x_1x_2x_3z_{21}z_{32} = x_1x_2z_{21}x_3z_{32} = x_2x_1x_2.x_3x_2. \\
111011 : g &= x_1x_2x_3z_{31}z_{32} = x_1z_{31}x_2x_3z_{32} = x_1x_3x_1.x_3x_1x_3.x_2. \\
110111 : g &= x_1x_2z_{21}z_{31}z_{32} = x_2x_1z_{31}z_{32} = z_{32}x_2x_1z_{31} = x_3x_2x_3.x_1x_3x_1x_3x_1. \\
101111 : g &= x_1x_3z_{21}z_{31}z_{32} = x_1x_3z_{31}z_{32}z_{21} = x_3x_1z_{32}z_{21} = x_3z_{32}z_{21}x_1 = x_2x_3x_2.x_2x_1x_2.
\end{aligned}$$

$$011111 : g = x_2 x_3 z_{21} z_{31} z_{32} = z_{21} z_{31} x_2 x_3 z_{32} = x_3 x_1 x_3 x_1 x_3 \cdot x_1 x_2 x_1.$$

Thus g is a product of at most three palindromes.

Let $|h| = 6$. Then the only possibility is 111111 and we have

$$g = x_1 x_2 x_3 z_{21} z_{31} z_{32} = x_1 x_2 z_{21} x_3 z_{31} z_{32} = x_2 \cdot x_3 x_1 x_3 \cdot x_2 x_3 x_2.$$

Thus we have shown that any g in $\overline{N}_{3,2}$ can be written as a product of at most four palindromes. Thus $pw(\overline{N}_{3,2}) \leq 4$. On the other hand we have seen at Lemma 3.7 that the element $h = z_{21} z_{31} z_{32}$ can not be written as a product of ≤ 3 palindromes. Hence $\overline{N}_{3,2}$ has at least one element whose palindromic length is at least 4. Thus $pw(\overline{N}_{3,2}) \geq 4$.

This proves that we must have $pw(\overline{N}_{3,2}) = 4$. \square

In view of Lemma 3.7, we have actually shown little more in the above proof:

Corollary 3.3. *In $\overline{N}_{3,2}$ the only element that can not be expressed as a product of three palindromes is $z_{21} z_{31} z_{32}$. Moreover, $l_P(z_{21} z_{31} z_{32}) = 4$.*

3.5. $r = 3$. We shall consider the 3-step two generator group $N_{2,3}$ in this subsection. In the groups $N_{n,3}$ the following commutator formulas hold

$$\begin{aligned} [y, x^r] &= [y, x]^r [[y, x], x]^{r(r-1)/2}, \\ [[y^r, x], z] &= [[y, x^r], z] = [[y, x]^r, z] = [[y, x], z]^r = [[y, x], z]^r. \end{aligned}$$

Proposition 3.3. $3 \leq pw(N_{2,3}) \leq 6$.

Proof. Any element $g \in N_{2,3}$ has the form $g = x^\alpha y^\beta d$ for some integers α, β and $d \in N'_{2,3}$. Since $N'_{2,3}$ is normal in $N_{2,3}$ then

$$g = x^\alpha d_1 y^\beta, \text{ where } d_1 = y^\beta d y^{-\beta}.$$

We assert that the element d_1 can be presented in the form

$$d_1 = [y, x, x]^a [x^b [y, x]^c, y]$$

for some integers a, b, c . Indeed, write the commutator $[x^b [y, x]^c, y]$ as a product of basis commutators. Using the commutator identities, we have

$$\begin{aligned} [x^b [y, x]^c, y] &= [x^b, y]^{[y, x]^c} [[y, x]^c, y] = [x^a, y] [[y, x], y]^c = \\ &= [x, y]^b [[x, y], x]^{b(b-1)/2} [[y, x], y]^c = [y, x]^{-b} [[y, x], x]^{-b(b-1)/2} [[y, x], y]^c. \end{aligned}$$

Since, any element in $N'_{2,3}$ is a product of the following basis commutators and their inverses:

$$[y, x], [y, x, x], [y, x, y],$$

our assertion holds. Hence, any element $g \in N_{2,3}$ is represented in the form

$$g = x^\alpha [y, x, x]^a [x^b [y, x]^c, y] y^\beta, \quad \alpha, \beta, a, b, c \in \mathbb{Z}.$$

Since, the commutator $[y, x, x]^a = [y, x, x^a]$ lies in the center of $N_{2,3}$, then

$$g = [y, x, x^a] x^\alpha [x^b [y, x]^c, y] y^\beta.$$

By Lemma 2.3(3) each of the elements $[y, x, x^a] x^\alpha$ and $[x^b [y, x]^c, y] y^\beta$ is the product of 3 palindromes. Hence, g is a product of 6 palindromes. This shows that any element in $N_{2,3}$ is a product of 6 palindromes. Hence, $pw(N_{2,3}) \leq 6$. It follows from Corollary 2.1

that $pw(N_{2,2}) \leq pw(N_{2,3})$, hence $pw(N_{2,3}) \geq 3$ by Proposition 3.1. This proves the result. \square

3.6. Proof of Theorem 1.1. Theorem 1.1 is obtained by combining Lemma 3.1, Lemma 3.2, Proposition 3.1, Proposition 3.2, Corollary 3.1 and Proposition 3.3.

3.7. Palindromic width of the free abelian - by - nilpotent groups.

Let $G = \langle x_1, x_2, \dots, x_n \rangle$ be a non-abelian free abelian - by - nilpotent group freely generated by x_1, x_2, \dots, x_n . Let A be an abelian normal subgroup of G such that G/A is nilpotent. For this group we have

Lemma 3.8. *If G is a non-abelian free abelian - by - nilpotent group then $pw(G) \leq 5n$.*

Proof. It follows from [2, Theorem 2] that any element $g \in G$ has the form

$$g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} [u_1, x_1]^{a_1} [u_2, x_2]^{a_2} \dots [u_n, x_n]^{a_n}, \quad \alpha_i \in \mathbb{Z}, u_i \in N_{n,2}, a_i \in A.$$

By (3) of Lemma 2.3 any commutator $[u_i, x_i]$ is a product of 3 palindromes, thus by (2) of Lemma 2.3, any commutator $[u_i, x_i]^{a_i}$ is a product of 4 palindromes. Hence, g is a product of $n + 4n = 5n$ palindromes. \square

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SOBOLEV INSTITUTE OF MATHEMATICS, NOVOSIBIRSK, 630090, RUSSIA
E-mail address: **bardakov@math.nsc.ru**

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH (IISER) MOHALI, KNOWLEDGE CITY, SECTOR 81, S.A.S. NAGAR, P.O. MANAULI 140306, INDIA
E-mail address: **krishnendu@iisermohali.ac.in**, **krishnendug@gmail.com**